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# A class of harmonic almost-product structures

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**Abstract**

The energy of a Riemannian almost-product structure $P$ is measured by forming the Dirichlet integral of the associated Gauss section $\gamma$, and $P$ is decreed harmonic if $\gamma$ criticalizes the energy functional when restricted to the submanifold of sections of the Grassmann bundle. Euler–Lagrange equations are obtained, and geometrically transformed in the special case when $P$ is totally geodesic. These are seen to generalize the Yang–Mills equations, and generalizations of the self-duality and anti-self-duality conditions are suggested. Several applications are then described. In particular, it is considered whether integrability of $P$ is a necessary condition for $\gamma$ to be harmonic.

*Key words:* harmonic section, Grassmann bundle, almost-product structure, totally geodesic, Nijenhuis tensor, Weitzenböck formula, Codazzi equation, Bianchi identity, Riemannian foliation

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## 0. Introduction

A *Riemannian almost-product (AP) structure* on a Riemannian $n$-manifold $(M, g)$ is an orthogonal $(1,1)$ tensor field $P$ on $M$ with $P^2 = 1$ and $P \neq \pm 1$; equivalently, a pair of non-trivial orthogonal complementary distributions $(\mathcal{F}, \mathcal{G})$ on $M$, the eigendistributions of $P$. If the rank of $\mathcal{F}$ is $k$, such a structure is parametrized by a section $\gamma$ of the Grassmann bundle $\pi : G_k M \to M$ of $k$-planes in $TM$: just define $\gamma(x) = \mathcal{F}_x$. When $M = \mathbb{R}^n$ and $\mathcal{F}$ is integrable, the restriction of $\gamma$ to any leaf of the corresponding foliation is the graph of the Gauss map for that leaf; we therefore refer to $\gamma$ as the *Gauss section* associated to $P$. Since $G_k M$ has a natural Riemannian metric relative to $g$ (viz. the direct sum of the (Levi-Civita) horizontal lift of $g$ with the metric induced on the fibres by the usual $O(n)$-invariant metric on the Grassmannian

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$G_k(\mathbb{R}^n)$), it is possible to measure the energy of $\gamma$, and seek critical points with respect to variations through sections. Such $\gamma$ are called *harmonic sections* [17]; the associated $P$ will therefore be called *harmonic AP-structures*. They are characterized by the following non-linear (quasi-linear) system of second order PDEs, generalizing the first order linear system $\nabla P = 0$:

$$[P, \nabla^* \nabla P] = 0 \tag{0.1}$$

where $\nabla^*\nabla$ denotes the rough Laplacian of $(M, g)$ and $[\,,\,]$ is the commutator bracket. Equations (0.1) are elliptic, provided $P$ satisfies the constraint equation $P^2 = 1$; they are derived in §1 below (see theorem 1.4).

In §2 we focus on the class of *totally geodesic* (t.g.) AP-structures, whose defining condition is that both $\mathcal{F}$ and $\mathcal{G}$ are t.g. plane fields i.e. all geodesics with initial vector in $\mathcal{F}$ (resp. $\mathcal{G}$) remain tangent to $\mathcal{F}$ (resp. $\mathcal{G}$) for all time. (It should be noted that this in no way relates to $\gamma$ being a t.g. map.) If $\mathcal{F}$ or $\mathcal{G}$ is integrable, we have a Riemannian foliation with t.g. leaves, examples of which include: foliations of Lie groups by translates of a fixed Lie subgroup; Riemannian submersions with t.g. fibres; the total space of a complete fibre bundle with connection, equipped with a Kaluza–Klein metric (see example 3.12). However, t.g. AP-structures which are non-integrable (in the sense of neither $\mathcal{F}$ nor $\mathcal{G}$ being integrable) are easily constructed. For example, an invariant AP-structure $P$ on a Lie group is t.g. with respect to any bi-invariant metric, and if neither $\mathcal{F}_e$ nor $\mathcal{G}_e$ is a subalgebra (where $e$ is the group identity), then $P$ is non-integrable. More generally, invariant AP-structures on a naturally reductive homogeneous Riemannian manifold are t.g. (see example 2.3). Non-integrable AP-structures appear in classical mechanics, as 'non-holonomic systems with ideal constraints' [2, p. 96]; for example, a ball rolling on an 'absolutely rough' plane (see example 3.10). In broader terms, t.g. AP-structures are analogous to almost-Kähler structures in Hermitian geometry. Part of this analogy is based on formal computations in the symmetric algebra of $M$, as opposed to its exterior algebra; see for example proposition 2.6.

The main purpose of this paper is to provide geometric characterizations of equations (0.1), and two are given in §3, in case $P$ is t.g. The first (theorem 3.1) involves the curvature tensor of $(M, g)$. When viewed alongside a curvature irreducibility result (theorem 2.2) it suggests that harmonic t.g. AP-structures are really rather strong generalizations of parallel AP-structures. The second involves the Nijenhuis tensor of $N$ of $P$ (see (2.2) for the definition), which is a $TM$-valued 2-form on $M$. The coderivative (or covariant divergence) $\delta N$ is therefore a field of endomorphisms of $M$, and we prove:

**Theorem 3.5.** *A t.g. AP-structure is harmonic if and only if $\delta N$ is self-adjoint.*

This characterization generalizes the Yang–Mills equations for fibre bundles, and for t.g. Riemannian foliations of codimension 4 suggests a generalization of the self-dual and anti-self-dual Yang–Mills equations (theorem 3.8). Instrumental to our geometrization procedure are generalizations to arbitrary AP-structures of Codazzi's equations for a submanifold (3.3) and Bianchi's identity for a principal bundle connection (3.7). Finally, we give some applications of our results. In 3.9 we consider invariant AP-structures $P$ on a Lie group with bi-invariant metric, and observe that $P$ is harmonic if $P_e$ is an *automorphism* of the Lie algebra. We also show the converse is false, by observing that on a compact semi-simple Lie group any invariant AP-structure is harmonic with respect to the Killing metric, provided $\mathcal{F}$ or $\mathcal{G}$ is integrable. Example 3.10 is an invariant AP-structure on the Lie group $SO(3) \times SO(3)$, representing the constraints in phase space of a sphere rolling on another 'absolutely rough' sphere. We show this AP-structure is harmonic precisely when the two spheres have equal radii, in which case $P_e$ is an automorphism; in fact this is the only case where either eigendistribution is integrable. Perhaps, in the context of Lie groups, integrability (of $\mathcal{F}$ or $\mathcal{G}$) is a necessary and sufficient condition for harmonicity? This question is resolved by example 3.11, which is a harmonic t.g. AP-structure on $SO(3) \times SO(3)$ with neither eigendistribution integrable. Example 3.12 is non-homogeneous; we consider the natural AP-structure on the total space of the tangent bundle of a Riemannian manifold. With respect to the Sasaki metric, this structure is harmonic if and only if the base manifold has harmonic curvature (cf. [19, Thm. 6.2]).

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**Conventions.** Our curvature convention is: $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. The summation convention is used throughout.

## 1. Harmonic AP-structures

Let $G = O(m)$, and let $\xi : O(M) \to M$ denote the principal $G$-bundle of orthonormal tangent frames of $(M, g)$. The Grassmann bundle $\pi : G_k M \to M$ may be constructed by factoring $\xi$ through $O(M)/H$ where $H = O(k) \times O(n-k)$; thus:

$$\begin{array}{ccc} O(M) & & \\ & \stackrel{\zeta}{\searrow} & \\ \xi \downarrow & & G_k M = O(M)/H \\ & \swarrow_{\pi} & \\ M = O(M)/G & & \end{array}$$

The quotient map $\zeta : O(M) \to G_kM$ is a principal $H$-bundle. We write $TG_kM = \mathcal{V} \oplus \mathcal{H}$ where $\mathcal{V} = \ker d\pi$ and $\mathcal{H}$ is the $\zeta$-image of the Levi-Civita horizontal distribution on $O(M)$. There is an induced splitting of the differential of any section $\gamma$, which we write:

$$d\gamma = d^v\gamma + d^h\gamma.$$

If $G_kM$ is equipped with the Riemannian metric described in §0, which we shall refer to as the *Kaluza–Klein metric*, then $\pi$ is a Riemannian submersion, and hence $|d^h\gamma|$ is constant. It therefore suffices to consider the *vertical energy functional*:

$$E^v(\gamma; U) = \tfrac{1}{2}\int_U |d^v\gamma|^2\, d\mu, \qquad U \subset M \text{ relatively compact}$$

where $d\mu$ is the Riemannian volume element. Moreover, since $\pi$ has t.g. fibres (cf. [16]), by [17] the Euler–Lagrange equations for a critical point of $E^v$ constrained to the submanifold of sections $\mathcal{C}(\pi)$ reduce to

$$\tau^v(\gamma) = \operatorname{Tr}\nabla^v d^v\gamma = 0 \tag{1.1}$$

where $\nabla^v$ is the $\mathcal{V}$-component of the Levi-Civita connection of the Kaluza–Klein metric. Harmonic map terminology [5] suggests that *vertical tension field* is the appropriate name for $\tau^v(\gamma)$. Thus, a harmonic AP-structure $P$ is one for which the vertical tension of $\gamma$ vanishes.

To achieve our aim in §1 of expressing (1.1) as an equation in $P$ (theorem 1.4), a more detailed description of the geometry of the Grassmann bundle is necessary. We note firstly the existence of a tautological AP-structure $\mathcal{P}$ in the pullback $\pi^*TM \to G_kM$; namely, if $y \in G_kM$ then $\mathcal{P}(y)$ is the involution of $T_{\pi(y)}M$ whose matrix with respect to any frame in $\zeta^{-1}(y)$ is

$$P_o = \begin{pmatrix} 1_k & 0 \\ 0 & -1_{n-k} \end{pmatrix}$$

We note also the existence of a canonical isometric vector bundle embedding $\iota : \mathcal{V} \hookrightarrow \pi^*\mathcal{E}$, where $\mathcal{E} \to M$ is the skew-symmetric subbundle of $\operatorname{End}(TM)$. The construction of $\iota$ goes as follows. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of $G$ and $H$ respectively, and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the usual decomposition, viz. orthogonal with respect to the Killing form. Elements of $\mathfrak{m}$ are skew-symmetric matrices which anticommute with $P_o$. The $\mathfrak{m}$-component of the Maurer–Cartan form of $G$ is $H$-equivariant and therefore projects to a non-degenerate bundle-valued 1-form on the Grassmannian $G/H$, which may be transferred fibre-by-fibre to $G_kM$. The image of $\iota$ is the vector bundle associated to $\zeta$ with fibre $\mathfrak{m}$, which will be denoted $\mathcal{E}_{\mathfrak{m}} \to G_kM$. It is characterized as the subbundle of $\pi^*\mathcal{E}$ whose elements anticommute with $\mathcal{P}$. Let $\kappa : TN \to \mathcal{E}_{\mathfrak{m}}$ denote the composition of $\iota$ with the horizontal projection of $TN$ onto $\mathcal{V}$.

**Lemma 1.1.** *For all $E \in TG_kM$ we have $\kappa(E) = -\frac{1}{2}\mathcal{P} \circ \nabla_E \mathcal{P}$, where the covariant derivative is the $\pi$-pullback of the Riemannian connection of $g$.*

*Proof.* Let $\omega$ be the $\mathfrak{g}$-valued Levi-Civita connection 1-form on $O(M)$. The component $\omega_{\mathfrak{m}}$ is $H$-equivariant, vanishes on $\ker d\zeta$, and its restriction to $\ker d\xi$ is the $\mathfrak{m}$-component of the Maurer–Cartan form. Therefore, the projection of $\omega_{\mathfrak{m}}$ to an $\mathcal{E}_{\mathfrak{m}}$-valued 1-form on $G_kM$ coincides with $\kappa$.

The bundle $\pi^*(\mathrm{End}\ TM) \to G_kM$ is associated to the $G$-extension $\pi^*O(M) \to G_kM$ of the principal $H$-bundle $\zeta$. Let $\tilde{\mathcal{P}} : \pi^*O(M) \to \mathfrak{gl}(m)$ denote the $G$-equivariant lift of the section $\mathcal{P}$; by definition $\tilde{\mathcal{P}}$ is the $G$-extension of $P_o$. If $D$ denotes the exterior covariant derivative for $\omega$, and $\tilde{E} \in TO(M)$ is any lift of $E$ then

$$D\tilde{\mathcal{P}}(\tilde{E}) = d\tilde{\mathcal{P}}(\tilde{E}) + [\omega(\tilde{E}), \tilde{\mathcal{P}}] = [\omega_{\mathfrak{m}}(\tilde{E}), \tilde{\mathcal{P}}] = -2\tilde{\mathcal{P}}.\omega_{\mathfrak{m}}(\tilde{E})$$

since $\tilde{\mathcal{P}}|O(M) = P_o$ and elements of $\mathfrak{m}$ anticommute with $P_o$. Projection to $G_kM$ yields

$$\nabla_E \mathcal{P} = -2\mathcal{P} \circ \kappa(E)$$

and the result follows since $\mathcal{P}^{-1} = \mathcal{P}$. □

**Proposition 1.2.** *If $\gamma \in \mathcal{C}(\pi)$ parametrizes the AP-structure $P$, then*

$$\iota(d^v\gamma(X)) = -\tfrac{1}{2} P \circ \nabla_X P, \qquad \forall X \in TM.$$

*Proof.* Since $P$ is the $\gamma$-pullback of the tautological AP-structure $\mathcal{P}$, and $\iota \circ d^v\gamma = \kappa \circ d\gamma$, the result follows on taking the $\gamma$-pullback of the lemma (using $\pi \circ \gamma = \mathrm{id}$). □

In order to characterize the vertical tension field, it is clear from (1.1) that we also need to compute the $\iota$-image of $\nabla^v$. The following formula is slightly more general.

**Lemma 1.3.** *If $E \in TG_kM$ and $F$ is a vector field on $G_kM$ then*

$$\kappa(\nabla_E F) = \tfrac{1}{2}\mathcal{P}[\mathcal{P}, \nabla_E(\kappa F)] - \tfrac{1}{4}\mathcal{P}[\mathcal{P}, R(\pi_* E, \pi_* F)]$$

*where the connection $\nabla$ and curvature tensor $R$ on the right hand side are those of $(M, g)$, and on the left hand side is the Levi-Civita connection of the Kaluza–Klein metric.*

*Proof.* Let $\langle , \rangle$ denote the Kaluza–Klein metric. If $L$ is a vertical vector field on $G_kM$, and $E$ is extended to a local vector field, then by [8, Ch. IV, Prop. 2.3]

$$2\langle \nabla_E F, L\rangle = E.\langle F, L\rangle + F.\langle E, L\rangle - L.\langle E, F\rangle$$

$$-\langle E,[F,L]\rangle - \langle F,[E,L]\rangle + \langle L,[E,F]\rangle.$$

Since $\kappa|\mathcal{V}$ is isometric and the restriction of $\langle,\rangle$ to $\mathcal{H}$ is the horizontal lift of $g$, it follows that $\langle E,F\rangle = g(\pi_* E,\pi_* F) + g(\kappa E,\kappa F)$ etc. and therefore

$$\begin{aligned} 2\,g(\kappa(\nabla_E F),\kappa L) &= E.g(\kappa F,\kappa L) + F.g(\kappa E,\kappa L) - L.g(\kappa E,\kappa F) \\ &\quad - g(\kappa E,\kappa[F,L]) - g(\kappa F,\kappa[E,L]) + g(\kappa L,\kappa[E,F]) \\ &\quad - L.g(\pi_* E,\pi_* F) - g(\pi_* E,\pi_*[F,L]) - g(\pi_* F,\pi_*[E,L]). \end{aligned}$$

We claim that each of the three terms involving $\pi_*$ vanishes. This is clearly so if at least one of $E,F$ is vertical. If both $E,F$ are horizontal, then since $\nabla_E F$ depends only on the values of $F$ on a slice transverse to the fibres of $\pi$ we may assume that both $E,F$ are $\pi$-projectible. The claim then follows from the fact that $L$ is $\pi$-adapted to the zero field on $M$. To expand the remaining terms, use the metric property of $\nabla$:

$$\begin{aligned} 2\,g(\kappa(\nabla_E F),\kappa L) &= g(2\nabla_E(\kappa F) - d\kappa(E,F),\kappa L) \\ &\quad + g(d\kappa(E,L),\kappa F) + g(d\kappa(F,L),\kappa E) \end{aligned}$$

where $d\kappa$ is the antisymmetrization of $\nabla\kappa$. Now the $\mathcal{E}_{\mathfrak{m}}$-component of $\nabla_E(\kappa F)$ is $\frac{1}{2}\mathcal{P}[\mathcal{P},\nabla_E(\kappa F)]$ and so

$$g(\nabla_E(\kappa F),\kappa L) = \tfrac{1}{2}\,g(\mathcal{P}[\mathcal{P},\nabla_E(\kappa F)],\kappa L)$$

Further, since $\kappa$ is the projection to $G_k M$ of $\omega_{\mathfrak{m}}$, the $\mathcal{E}_{\mathfrak{m}}$-component of $d\kappa$ is the projection of the horizontal component of $d\omega_{\mathfrak{m}}$. The $\mathfrak{m}$-component of the Structure eq. is

$$d\omega_{\mathfrak{m}} = \Omega_{\mathfrak{m}} - [\omega,\omega]_{\mathfrak{m}}$$

where $\Omega$ is the Levi-Civita curvature 2-form. Because $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$, the horizontal component of $[\omega,\omega]_{\mathfrak{m}}$ vanishes. Since $\Omega_{\mathfrak{m}}$ is horizontal, it follows that the $\mathcal{E}_{\mathfrak{m}}$-component of $d\kappa$ coincides with the $\mathcal{E}_{\mathfrak{m}}$-component of the $\pi^*\mathcal{E}$-valued 2-form $\pi^* R$:

$$g(d\kappa(E,F),\kappa L) = \tfrac{1}{2}\,g(\mathcal{P}[\mathcal{P},R(\pi_* E,\pi_* F)],\kappa L) \qquad \text{etc.}$$

In particular $g(d\kappa(E,L),\kappa F) = 0 = g(d\kappa(F,L),\kappa E)$ since $L$ is vertical, and the proof of the lemma is complete. □

It is now possible characterize the vertical tension field. We define $\tau(P) = \imath(\tau^v(\gamma))$, which it is reasonable to call the *tension field of P*.

**Theorem 1.4.** *If P is any Riemannian AP-structure then* $\tau(P) = \frac{1}{4}[P,\nabla^*\nabla P]$ *where* $\nabla^*\nabla P = -\operatorname{Tr}\nabla^2 P$.

*Proof.* Since $R$ is skew-symmetric, it follows from (1.1) and lemma 1.3 (pulled-back by $\gamma$) that

$$\tau(P) = \mathrm{Tr}(\iota \circ \nabla^v d^v \gamma) = -\tfrac{1}{2}\,\mathrm{Tr}\,P[P, \nabla(\iota \circ d^v \gamma)]$$

Now by proposition 1.2

$$\tau(P) = -\tfrac{1}{4}\,\mathrm{Tr}\,P[P, \nabla(P \circ \nabla P)] = -\tfrac{1}{4}\,\mathrm{Tr}\,P[P, (\nabla P)^2 + P \circ \nabla^2 P]$$
$$= \tfrac{1}{4}\,[P, \nabla^* \nabla P]$$

since $(\nabla P)^2$ commutes with $P$. □

It follows from 1.4 that $P$ is harmonic precisely when $[P, \nabla^*\nabla P] = 0$. We note that this equation was obtained by G.Valli as the condition for the loop of gauge transformations determined by $P$ to be a closed geodesic [15].

## 2. Totally geodesic AP-structures

To any Riemannian AP-structure may be associated the following tensor field of type (2,1):

$$\alpha(X,Y) = \tfrac{1}{4}\,(\nabla_X P(PY) + \nabla_{PX} P(Y)) \tag{2.1}$$

called the *(total) second fundamental form*, which vanishes precisely when $P$ is parallel. Let $\alpha = S + N$ denote the symmetric/antisymmetric decomposition, where $S$ is the *symmetric second fundamental form* [13]:

$$S(X,Y) = \tfrac{1}{8}\,(\nabla_X P(PY) + \nabla_Y P(PX) + \nabla_{PX} P(Y) + \nabla_{PY} P(X))$$

and $N$ is the *Nijenhuis tensor* [11]:

$$N(X,Y) = \tfrac{1}{8}([X,Y] + [PX,PY] - P[PX,Y] - P[X,PY]). \tag{2.2}$$

The t.g. AP-structures are precisely those with $S \equiv 0$.

Let $\mathcal{F}$ (resp. $\mathcal{G}$) be the eigendistribution of $P$ with eigenvalue 1 (resp. $-1$), and let $p = \frac{1}{2}(1 + P) : TM \to \mathcal{F}$ and $q = \frac{1}{2}(1 - P) : TM \to \mathcal{G}$ be the projections. We reserve $U, V, W$ (resp. $A, B, C$) to denote elements or local sections of $\mathcal{F}$ (resp. $\mathcal{G}$); arbitrary tangent vectors or vector fields will continue to be denoted by $X, Y, Z$. Local orthonormal frame fields on $M$ will be denoted $(E_i\colon 1 \leqslant i \leqslant n)$; local orthonormal framings of $\mathcal{F}$ and $\mathcal{G}$ will be denoted by $(E_u\colon 1 \leqslant u \leqslant k)$ and $(E_a\colon k+1 \leqslant a \leqslant n)$ respectively. We write $\alpha|\mathcal{F}\times\mathcal{F} = \alpha_{\mathcal{F}}$ and $\alpha|\mathcal{G}\times\mathcal{G} = \alpha_{\mathcal{G}}$, noting that $\alpha|(\mathcal{F}\times\mathcal{G}) \oplus (\mathcal{G}\times\mathcal{F}) = 0$. Then

$$\alpha_{\mathcal{F}}(U,V) = q(\nabla_U V) \quad\text{and}\quad \alpha_{\mathcal{G}}(A,B) = p(\nabla_A B).$$

It follows that

$$S_{\mathcal{F}}(U,V) = \tfrac{1}{2}q(\nabla_U V + \nabla_V U)$$
$$S_{\mathcal{G}}(A,B) = \tfrac{1}{2}p(\nabla_A B + \nabla_B A). \tag{2.3}$$

Furthermore

$$N_{\mathcal{F}}(U,V) = \tfrac{1}{2}q[U,V] \quad \text{and} \quad N_{\mathcal{G}}(A,B) = \tfrac{1}{2}p[A,B]$$

are the integrability tensors for $\mathcal{F}$ and $\mathcal{G}$ respectively. The vector fields

$$H_{\mathcal{F}} = \operatorname{Tr}\alpha_{\mathcal{F}} \quad \text{and} \quad H_{\mathcal{G}} = \operatorname{Tr}\alpha_{\mathcal{G}}$$

are the *mean curvatures* of $\mathcal{F}$ and $\mathcal{G}$ respectively.

We firstly show that the existence of a t.g. AP-structure imposes certain restrictions on $(M,g)$, and derive a curvature irreducibility result analogous to [6, Cor. 4.3] for almost-Kähler structures. The source of both is a curvature identity generalizing [12, Thm. 3] for Riemannian submersions to the situation where neither of $\mathcal{F},\mathcal{G}$ is integrable. Let $\overline{\nabla}$ denote the projection of the Levi-Civita connection into either of the vector bundles $\mathcal{F},\mathcal{G} \to M$ (the context will indicate which), and also the appropriate extension to tensor products; for example, $\alpha_{\mathcal{F}}$ is a section of $\mathcal{F}^* \otimes \mathcal{F}^* \otimes \mathcal{G}$ and we write

$$\overline{\nabla}_X\alpha_{\mathcal{F}}(U,V) = \overline{\nabla}_X(\alpha_{\mathcal{F}}(U,V)) - \alpha_{\mathcal{F}}(\overline{\nabla}_X U, V) - \alpha_{\mathcal{F}}(U, \overline{\nabla}_X V).$$

Furthermore, if $U \in \mathcal{F}_x$ it will be convenient to denote by $\alpha_{\mathcal{F},U} : T_xM \to T_xM$ the self-adjoint extension of the endomorphism defined on $\mathcal{F}_x$ by $\alpha_{\mathcal{F},U}(V) = \alpha_{\mathcal{F}}(U,V)$; thus:

$$\alpha_{\mathcal{F},U}|\mathcal{G} : \mathcal{G} \to \mathcal{F};\ \alpha_{\mathcal{F},U}(A) = -p(\nabla_U A).$$

**Lemma 2.1.** *The following identity holds for any Riemannian AP-structure:*

$$\begin{aligned} g(R(U,A)V,B) = &-g(\overline{\nabla}_A\alpha_{\mathcal{F}}(U,V),B) - g(V,\overline{\nabla}_U\alpha_{\mathcal{G}}(A,B)) \\ &+ g(\alpha_{\mathcal{F},U}(A),(S_{\mathcal{F},V} - N_{\mathcal{F},V})B) \\ &+ g(\alpha_{\mathcal{G},A}(U),(S_{\mathcal{G},B} - N_{\mathcal{G},B})V). \end{aligned}$$

*Proof.* Summarizing the calculations, contributions to $R(U,A)V$ are made as follows:

$$\begin{aligned} g(\nabla_U\nabla_A V,B) &= -g(V,\overline{\nabla}_U\alpha_{\mathcal{G}}(A,B)), \\ g(\nabla_A\nabla_U V,B) &= g(\overline{\nabla}_A\alpha_{\mathcal{F}}(U,V),B), \\ g(\nabla_{[U,A]}V,B) &= g((N_{\mathcal{F},V} - S_{\mathcal{F},V})\circ\alpha_{\mathcal{F},U}(A),B) \\ &\quad + g(V,\ (N_{\mathcal{G},B} - S_{\mathcal{G},B})\circ\alpha_{\mathcal{G},A}(U)) \end{aligned}$$ □

**Theorem 2.2.** (1) *A Riemannian manifold $(M,g)$ admits a t.g. AP-structure only if not all sectional curvatures of $(M,g)$ are negative. If all sectional* *curvatures are strictly positive then at most one of $\mathcal{F}, \mathcal{G}$ is integrable.* (2) *If $P$ is a t.g. AP-structure and $[R, P] = 0$ then $P$ is parallel.*

*Proof.* If $S \equiv 0$ then the lemma implies

$$|U|^2 |A|^2 K(U \wedge A) = g(R(U, A)A, U) = |N_{\mathcal{F},U}(A)|^2 + |N_{\mathcal{G},A}(U)|^2$$

where $K(U \wedge A)$ is the sectional curvature of the 2-plane spanned by $U$ and $A$. This proves (1). If in addition $[R, P] = 0$ then $R(X, Y)$ leaves invariant the eigendistributions of $P$; in particular, each $K(U \wedge A)$ vanishes. Thus $N \equiv 0$, and hence $\alpha \equiv 0$. □

**Example 2.3.** Let $M$ be a Lie group with a bi-invariant metric $g$ (for example, $M$ compact). The Levi-Civita connection is then characterized on left-invariant vector fields by [7, p. 148]

$$\nabla_X Y = \tfrac{1}{2} [X, Y]$$

It follows immediately from (2.3) that any invariant Riemannian AP-structure is t.g.

More generally, suppose $(M, g)$ is a naturally reductive homogeneous Riemannian manifold, relative to a subgroup $K$ of isometries. Then any $K$-invariant AP-structure is t.g. For, a characterization of such $(M, g)$ is that geodesics coincide with orbits of 1-parameter subgroups of $K$ [1]; hence a $K$-invariant plane field is automatically t.g. It is also well-known [14] that all sectional curvatures of $(M, g)$ are non-negative.

For AP-structures, the analogue of the Kähler 2-form in Hermitian geometry is the following quadratic differential:

$$\rho(X, Y) = g(PX, Y).$$

The symmetric algebra $\mathfrak{S}^*M$ of $(M, g)$ may be equipped with operators $d_s : \mathfrak{S}^{r-1}M \to \mathfrak{S}^r M$ and $\delta_s : \mathfrak{S}^{r+1}M \to \mathfrak{S}^r M$ where

$$d_s\lambda(X_1, \cdots, X_r) = \frac{1}{(r-1)!} \sum_{\sigma \in S_r} \nabla_{X_{\sigma(1)}} \lambda(X_{\sigma(2)}, \cdots, X_{\sigma(r)})$$

$$\delta_s\lambda(X_1, \cdots, X_r) = -\nabla_{E_i} \lambda(E_i, X_1, \cdots, X_r)$$

In particular, the following result shows that the 3-form $d_s\rho$ encodes the same information as the symmetric second fundamental form $S$.

**Proposition 2.4.** *A Riemannian AP-structure is t.g. if and only if $d_s\rho = 0$. Precisely:*

*(a)* $d_s\rho(\mathcal{F}, \mathcal{F}, \mathcal{F}) = 0 = d_s\rho(\mathcal{G}, \mathcal{G}, \mathcal{G})$;

(b) $d_s\rho\,(\mathcal{F},\mathcal{F},\mathcal{G}) = 0$ *if and only if* $S_{\mathcal{F}} = 0$;

(c) $d_s\rho\,(\mathcal{F},\mathcal{G},\mathcal{G}) = 0$ *if and only if* $S_{\mathcal{G}} = 0$.

*Proof.* We have

$$\begin{aligned} d_s\rho\,(X,Y,Z) &= \nabla_X\rho(Y,Z) + \nabla_Y\rho(Z,X) + \nabla_Z\rho(X,Y) \\ &= g(\nabla_X P(Y),Z) + g(\nabla_Y P(Z),X) + g(\nabla_Z P(X),Y). \end{aligned}$$

But $\nabla P|\mathcal{F}\times\mathcal{F} = 2\alpha_{\mathcal{F}}$ and $\nabla P|\mathcal{G}\times\mathcal{G} = -2\alpha_{\mathcal{G}}$ from which (a) follows immediately. Furthermore

$$\begin{aligned} d_s\rho\,(U,V,A) &= g(\nabla_U P(V) + \nabla_V P(U),A) \\ &= 4\,g(\alpha_{\mathcal{F}}(U,V) + \alpha_{\mathcal{F}}(V,U),A) = 8\,g(S_{\mathcal{F}}(U,V),A) \end{aligned}$$

from which (b) follows; the verification of (c) goes similarly. □

The symmetric Laplacian $\Delta_s$ is defined

$$\Delta_s\lambda = \delta_s\,d_s\lambda - d_s\,\delta_s\lambda$$

This operator is symmetric on compactly-supported forms, but not positive; however, the minus sign guarantees a Weitzenböck-type formula. We look only at $\lambda \in \mathfrak{S}^2 M$, in which case an associated symmetric endomorphism field $L$ is defined $g(LX,Y) = \lambda(X,Y)$. Let Ric denote the Ricci curvature of $(M,g)$, and let $\mathrm{Ric}_L$ denote the following symmetric 2-covariant tensor field:

$$\mathrm{Ric}_L(X,Y) = g(R(X,E_i)LE_i,Y).$$

**Theorem 2.5. Weitzenböck Formula for Quadratic Differentials.** *If* $\lambda \in \mathfrak{S}^2 M$ *then* $\Delta_s\lambda = \nabla^*\nabla\lambda - \Gamma(\lambda)$ *where*

$$\Gamma(\lambda)\,(X,Y) = \mathrm{Ric}(LX,Y) + \mathrm{Ric}(X,LY) - 2\,\mathrm{Ric}_L(X,Y).$$

*Proof.* A computation yields:

$$\begin{aligned} \delta_s\,d_s\lambda\,(X,Y) &= \nabla^*\nabla\lambda\,(X,Y) - \nabla^2_{E_i,X}\lambda\,(E_i,Y) - \nabla^2_{E_i,Y}\lambda\,(E_i,X) \\ d_s\,\delta_s\lambda\,(X,Y) &= -\nabla^2_{X,E_i}\lambda\,(E_i,Y) - \nabla^2_{Y,E_i}\lambda\,(E_i,X). \end{aligned}$$

Let us define

$$\begin{aligned} R(X,Y)\lambda &= \nabla_X\nabla_Y\lambda - \nabla_Y\nabla_X\lambda - \nabla_{[X,Y]}\lambda \\ \mathrm{Ric}_\lambda(X,Y) &= \big(R(X,E_i)\lambda\big)(E_i,Y) = \mathrm{Ric}_L(X,Y) - \mathrm{Ric}(X,LY). \end{aligned}$$

From the Ricci identity $R(X,Y)\lambda = \nabla^2_{X,Y}\lambda - \nabla^2_{Y,X}\lambda$ it then follows that

$$\Delta_s\lambda\,(X,Y) = \nabla^*\nabla\lambda\,(X,Y) + \mathrm{Ric}_\lambda(X,Y) + \mathrm{Ric}_\lambda(Y,X).$$ □

**Remark.** The Hodge–de Rham Laplacian was extended to an operator $\Delta_{\mathrm{Lic}}$ on the entire covariant tensor algebra by A. Lichnerowicz [10], [4, Ch. 1, I]. For $\lambda \in \mathfrak{S}^2 M$ the definition is $\Delta_{\mathrm{Lic}}\lambda = \nabla^*\nabla\lambda + \Gamma(\lambda)$.

Our analogy between t.g. AP-structures and almost-Kähler structures concludes with the following property. In contrast to the almost-Kähler case, when $M$ is closed there is no converse, because $\Delta_s$ is not positive.

**Proposition 2.6.** *If $P$ is t.g. then $\rho$ is harmonic ($\Delta_s\rho = 0$).*

*Proof.* For an arbitrary AP-structure we have

$$d_s\rho\,(X, PX, PY) = -g(\nabla_X P(X) + \nabla_{PX} P(PX), Y)$$

and hence

$$\delta_s\rho\,(Y) = \tfrac{1}{2}\, d_s\rho\,(E_i, PE_i, PY).$$

It follows from 2.4 that if $P$ is t.g. then $d_s\rho = 0 = \delta_s\rho$ and hence $\Delta_s\rho = 0$. □

## 3. Harmonic totally geodesic AP-structures

Weitzenböck Formula 2.5 may be applied to the quadratic differential $\rho$ associated to a Riemannian AP-structure $P$. From theorem 1.4 it then follows that

$$\tau(P) = \tfrac{1}{2}[\,\mathrm{Ric}_P - \tfrac{1}{2}\Delta_s\rho, P\,].$$

We introduce the *partial Ricci curvatures* determined by $P$:

$$\mathrm{Ric}_{\mathcal{F}}(X,Y) = g(R(X,E_u)E_u, Y)$$
$$\mathrm{Ric}_{\mathcal{G}}(X,Y) = g(R(X,E_a)E_a, Y)$$

in terms of which

$$\mathrm{Ric} = \mathrm{Ric}_{\mathcal{F}} + \mathrm{Ric}_{\mathcal{G}} \quad \text{and} \quad \mathrm{Ric}_P = \mathrm{Ric}_{\mathcal{F}} - \mathrm{Ric}_{\mathcal{G}}$$

The following result is now an immediate consequence of proposition 2.6.

**Theorem 3.1.** *If $P$ is a t.g. AP-structure then $\tau(P) = \frac{1}{2}[\,\mathrm{Ric}_P, P\,]$. Thus $P$ is harmonic precisely when any of the following equivalent curvature conditions hold:*

*(1)* $[\,\mathrm{Ric}_P, P\,] = 0$;

*(2)* $\mathrm{Ric}_P(\mathcal{F}, \mathcal{G}) = 0$;

*(3)* $\mathrm{Ric}_{\mathcal{F}}(\mathcal{F}, \mathcal{G}) = \mathrm{Ric}_{\mathcal{G}}(\mathcal{F}, \mathcal{G})$.

**Corollary 3.2.** (See also [18, corollary 2.19].) *If $P$ defines a t.g. Riemannian foliation, then $\tau(P) = \mp\frac{1}{2}[\mathrm{Ric}, P]$, the sign depending on whether the $\pm 1$ eigendistribution is integrable. $P$ is harmonic precisely when either of the following equivalent conditions holds:*

*(1)* $[\mathrm{Ric}, P] = 0$;

*(2)* $\mathrm{Ric}(\mathcal{F}, \mathcal{G}) = 0$.

*In particular, if $(M, g)$ is Einstein then $P$ is harmonic.*

*Proof.* If $\mathcal{F}$ is integrable, then Codazzi's eq. [8, Ch. VII, Prop. 4.3] applied to the leaves yields $\mathrm{Ric}_{\mathcal{F}}(\mathcal{F}, \mathcal{G}) = 0$; otherwise said, $\mathrm{Ric}_P(\mathcal{F}, \mathcal{G}) = -\mathrm{Ric}(\mathcal{F}, \mathcal{G})$. Similarly, if $\mathcal{G}$ is integrable then $\mathrm{Ric}_P(\mathcal{F}, \mathcal{G}) = \mathrm{Ric}(\mathcal{F}, \mathcal{G})$. The result then follows from 3.1. □

In the light of proposition 2.2(2), the condition $[\mathrm{Ric}, P] = 0$ is probably the strongest curvature invariance that could be expected for a t.g. Riemannian foliation with non-integrable normal bundle. The Gauss sections of t.g. AP-structures with $[R, P] = 0$ are all zeroes of the vertical energy functional (see proposition 1.2).

The proof of 3.2 used Codazzi's eq. to compute the off-diagonal component of the partial Ricci curvatures. When neither of $(\mathcal{F}, \mathcal{G})$ is integrable this is no longer valid; however, it is possible to generalize Codazzi's Equation. The proof is similar to that of lemma 2.1 and we omit the details.

**Theorem 3.3. Generalized Codazzi Equation.** *For any Riemannian AP-structure we have*

$$g(R(U,V)W,A) = g(\overline{\nabla}_U \alpha_{\mathcal{F}}(V,W) - \overline{\nabla}_V \alpha_{\mathcal{F}}(U,W), A) + 2\,g\big(W, \alpha(N(U,V),A)\big)$$

*and an analogous equation for $g(R(A,B)C,U)$.*

In the t.g. case, use of 3.3 to compute the partial Ricci curvatures will yield expressions involving the coderivatives $\overline{\delta}N_{\mathcal{F}}$ (a $\mathcal{G}$-valued 1-form on $\mathcal{F}$) and $\overline{\delta}N_{\mathcal{G}}$ (an $\mathcal{F}$-valued 1-form on $\mathcal{G}$). This suggests looking at the full coderivative $\delta N$. It is convenient to extend our notation as follows:

$$\alpha_{\mathcal{F},B} : \mathcal{F} \to \mathcal{F};\ U \mapsto \alpha_{\mathcal{F},U}(B) \quad \text{and} \quad \alpha_{\mathcal{G},U} : \mathcal{G} \to \mathcal{G};\ A \mapsto \alpha_{\mathcal{G},A}(U).$$

**Lemma 3.4.** *The components of $\delta N$ are:*

*(1)* $g(\delta N(U), A) = g\big(\overline{\delta}N_{\mathcal{F}}(U) + N(H_{\mathcal{G}}, U), A\big)$

*(2)* $g\big(U, \delta N(A)\big) = g\big(U, \overline{\delta}N_{\mathcal{G}}(A) + N(H_{\mathcal{F}}, A)\big)$

*(3)* $g(\delta N(U), V) = \frac{1}{2} g(N_{\mathcal{F},U}, N_{\mathcal{F},V} - S_{\mathcal{F},V}) - g(\alpha_{\mathcal{G},U}, N_{\mathcal{G},V})$

*(4)* $g\big(A, \delta N(B)\big) = \frac{1}{2} g(N_{\mathcal{G},A} - S_{\mathcal{G},A}, N_{\mathcal{G},B}) - g(N_{\mathcal{F},A}, \alpha_{\mathcal{F},B})$.

*Proof.* A routine calculation. □

**Theorem 3.5.** *If $P$ is a t.g. AP-structure then $\tau(P) = \frac{1}{2}[(\delta N)^\dagger - \delta N, P]$ where $(\delta N)^\dagger$ is the g-adjoint. The following equivalent integrability conditions are necessary and sufficient for $P$ to be harmonic:*

*(1) $\delta N$ is a self-adjoint endomorphism field;*

*(2) $g(\bar{\delta} N_{\mathcal{F}}(U), A) = g(U, \bar{\delta} N_{\mathcal{G}}(A))$.*

*Proof.* When $S \equiv 0$ the Generalized Codazzi Equation implies

$$\tfrac{1}{2}[\mathrm{Ric}_{\mathcal{F}}, P](U,A) = \mathrm{Ric}_{\mathcal{F}}(U,A) = -g(\bar{\delta} N_{\mathcal{F}}(U), A) - g(N_{\mathcal{F},U}, N_{\mathcal{G},A})$$
$$\tfrac{1}{2}[\mathrm{Ric}_{\mathcal{G}}, P](U,A) = \mathrm{Ric}_{\mathcal{G}}(U,A) = -g(U, \bar{\delta} N_{\mathcal{G}}(A)) - g(N_{\mathcal{F},U}, N_{\mathcal{G},A}).$$

It follows from theorem 3.1 that $\tau(P)$ is the difference of these two expressions:

$$g(\tau(P)U,A) = \tfrac{1}{2}[\mathrm{Ric}_P, P](U,A) = g(U, \bar{\delta} N_{\mathcal{G}}(A)) - g(\bar{\delta} N_{\mathcal{F}}(U), A)$$

and criterion (2) is immediate. From lemma 3.4 it follows that

$$\begin{aligned} g(\tau(P)U,A) &= g(U, \delta N(A)) - g(\delta N(U), A) \\ &= g(((\delta N)^\dagger - \delta N)U, A) = \tfrac{1}{2} g([(\delta N)^\dagger - \delta N, P]U, A) \end{aligned}$$

which establishes the formula for $\tau(P)$. Criterion (1) follows by noting from Lemma 3.4 that $\delta N$ is already symmetric on $\mathcal{F} \times \mathcal{F}$ and $\mathcal{G} \times \mathcal{G}$ provided $S \equiv 0$. □

**Corollary 3.6.** *If $P$ defines a t.g. Riemannian foliation, with leaves tangent to $\mathcal{F}$, then $P$ is harmonic if and only if $\bar{\delta} N_{\mathcal{G}} = 0$.*

When the t.g. Riemannian foliation is the total space of a principal fibre bundle with connection, then $N_{\mathcal{G}}$ is the curvature 2-form (see [8, Ch. II, Cor. 5.3]), and $\bar{\delta} N_{\mathcal{G}} = 0$ are the Yang–Mills equations. The special case of self-dual, or anti-self-dual connections may be generalized as follows. For an arbitrary Riemannian AP-structure, let $\mathfrak{A}^*(\mathcal{G})$ denote the exterior algebra of $\mathcal{G}$. If $\mathcal{G}$ is orientable there is a Hodge star operator

$$\bar{*} : \mathfrak{A}^r(\mathcal{G}) \otimes \mathcal{F} \to \mathfrak{A}^{n-k-r}(\mathcal{G}) \otimes \mathcal{F} \quad (0 \leqslant r \leqslant n-k)$$

defined in terms of the volume element of $\mathcal{G}$. There is then the characterization:

$$\bar{\delta} = (-1)^{(n-k)(r+1)+1} \bar{*} \bar{d} \bar{*} \tag{3.1}$$

If $n-k = 4$ then $\mathcal{G}$ may be designated *self-dual* or *anti-self-dual* according as

$$\bar{*} N_{\mathcal{G}} = N_{\mathcal{G}} \quad \text{or} \quad \bar{*} N_{\mathcal{G}} = -N_{\mathcal{G}} \tag{3.2}$$

The significance of ± self-duality in Yang–Mills theory depends on Bianchi's Identity, which generalizes to AP-structures as follows.

**Theorem 3.7. Generalized Bianchi Identity.** *For any Riemannian AP-structure we have*

$$g(\overline{d}N_{\mathcal{F}}(U,V,W),A) = -\mathfrak{C}\, g\big(U,\alpha(N(V,W),A)\big)$$

*where* $\mathfrak{C}$ *denotes the cyclic sum over* $U, V, W$. *There is an analogous identity for* $\overline{d}N_{\mathcal{G}}$.

*Proof.* By Bianchi's First Identity for $R$, the cyclic sum over $U, V, W$ in the Generalized Codazzi Equation 3.3 yields

$$0 = \mathfrak{C}\, g\big(\overline{\nabla}_U\alpha_{\mathcal{F}}(V,W) - \overline{\nabla}_U\alpha_{\mathcal{F}}(W,V),A\big) + 2\,\mathfrak{C}\, g\big(U,\alpha(N(V,W),A)\big)$$

Since $\overline{\nabla}_U S_{\mathcal{F}}$ is symmetric and $\overline{\nabla}_U N_{\mathcal{F}}$ is skew-symmetric, it follows that

$$\mathfrak{C}\,\big(\overline{\nabla}_U\alpha_{\mathcal{F}}(V,W) - \overline{\nabla}_U\alpha_{\mathcal{F}}(W,V)\big)$$
$$= 2\,\mathfrak{C}\,\big(\overline{\nabla}_U N_{\mathcal{F}}(V,W)\big) = 2\,\overline{d}N_{\mathcal{F}}(U,V,W). \qquad \square$$

**Theorem 3.8.** *Suppose P defines a t.g. Riemannian foliation of codimension 4. If the normal bundle is orientable and ± self-dual, then P is harmonic.*

*Proof.* From (3.1) and (3.2) it follows that

$$\overline{\delta}N_{\mathcal{G}} = -\bar{*}\,\overline{d}\,\bar{*}\,N_{\mathcal{G}} = \mp\bar{*}\,\overline{d}N_{\mathcal{G}}$$

But when $N_{\mathcal{F}} \equiv 0$ the Generalized Bianchi Identity (for $N_{\mathcal{G}}$) reduces to $\overline{d}N_{\mathcal{G}} = 0$, and the result follows from 3.6. $\square$

**Remark.** From proposition 1.2 and (2.1) it follows that $|d^v\gamma|^2 = 2|\alpha|^2$. Therefore if $P$ defines a t.g. Riemannian foliation then $|\alpha|^2 = |N_{\mathcal{G}}|^2$, and in codimension 4 we can write

$$E^v(\gamma,U) = \int_U |N_{\mathcal{G}}^+|^2 d\mu + \int_U |N_{\mathcal{G}}^-|^2 d\mu.$$

By analogy with Yang–Mills theory, one would like to express the difference of the two integrals on the right hand side as a characteristic number of the normal bundle, or some other topological constant. However, apart from the special case of a fibre bundle with connection, it is not obvious how this could be done, leaving us unable to infer in general that ± self-duality of the normal bundle implies minimum energy of the Gauss section.

**Example 3.9.** Let $(M,g)$ be a Lie group with bi-invariant metric. If $P$ is also bi-invariant (i.e. $P_e$ is Ad-equivariant) then

$$[\mathcal{F},\mathcal{F}] \subset \mathcal{F}, \quad [\mathcal{F},\mathcal{G}] = 0, \quad [\mathcal{G},\mathcal{G}] \subset \mathcal{G}$$

from which it follows that $\alpha \equiv 0$, and hence $\nabla P = 0$. Conversely, if $P$ is invariant and parallel, and $M$ connected, then $P$ is bi-invariant. So by (0.1) harmonic AP-structures on Lie groups generalize bi-invariant AP-structures.

If $P$ is invariant, and $P_e$ is a Lie algebra automorphism, then [8, Ch. XI, Prop. 2.1]

$$[\mathcal{F},\mathcal{F}] \subset \mathcal{F}, \qquad [\mathcal{F},\mathcal{G}] \subset \mathcal{G}, \qquad [\mathcal{G},\mathcal{G}] \subset \mathcal{F}. \tag{3.3}$$

In particular, $\mathcal{F}$ is integrable. The curvature tensor is $R(X,Y) = -\frac{1}{4}\,\mathrm{ad}[X,Y]$ (see [7, p. 148]) and therefore

$$\begin{aligned} \mathrm{Ric}_{\mathcal{F}}(X,Y) &= \tfrac{1}{4}\, g([X,E_u],[Y,E_u]), \\ \mathrm{Ric}_{\mathcal{G}}(X,Y) &= \tfrac{1}{4}\, g([X,E_a],[Y,E_a]). \end{aligned} \tag{3.4}$$

It follows from (3.3) and (3.4) that $\mathrm{Ric}(U,A) = 0$, and hence from example 2.3 and theorem 3.2 (2) that $P$ is harmonic.

Finally, suppose $M$ is compact semi-simple, and $g$ is the Killing metric. Then $(M,g)$ is an Einstein manifold [8, Ch. X, Ex. 3.2]. Therefore by theorem 3.2 any invariant $P$ with $\mathcal{F}$ or $\mathcal{G}$ integrable is harmonic. We note that such $P$ are not necessarily Lie algebra automorphisms.

**Example 3.10.** Let $S_1$ be a sphere of radius $s$, touching a sphere $S_2$ of radius $t$. Assume the centre of each $S_i$ is fixed, say on the $z$-axis, and the spheres are otherwise free to rotate. The configuration space may be identified with $M = SO(3) \times SO(3)$, equipped with the direct sum of the following multiples of the Killing metric:

$$\langle X,Y\rangle_1 = -\tfrac{1}{2}s^2\,\mathrm{Tr}(XY), \qquad \langle X,Y\rangle_2 = -\tfrac{1}{2}t^2\,\mathrm{Tr}(XY).$$

Let $r = s/t$. If the $S_i$ are assumed 'absolutely rough', then rotating $S_1$ forces $S_2$ to rotate in the following ways:

| rotation of $S_1$ | rotation of $S_2$ |
|---|---|
| $\theta$ about $x$-axis | $-r\theta$ about $x$-axis |
| $\theta$ about $y$-axis | $-r\theta$ about $y$-axis |
| $\theta$ about $z$-axis | $\theta$ about $z$-axis |

These constraints generate the following subspace of the Lie algebra:

$$\mathcal{F}_e = \{(ue_1 + ve_2 + we_3, ue_1 - rve_2 - rwe_3) : u,v,w \in \mathbb{R}\}$$

where $(e_1,e_2,e_3)$ is the standard basis of $\mathfrak{so}(3)$:

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{3.5}$$

The orthogonal complement is

$$\mathcal{G}_e = \{(ae_1 + be_2 + ce_3, -r^2ae_1 + rbe_2 + rce_3) : a, b, c \in \mathbb{R}\}$$

Orthonormal bases $(E_1, E_2, E_3)$ and $(E_4, E_5, E_6)$ of $\mathcal{F}_e$ and $\mathcal{G}_e$ respectively are as follows:

$$E_1 = \frac{1}{\sqrt{s^2+t^2}}(e_1, e_1), \quad E_2 = \frac{1}{s\sqrt{2}}(e_2, -re_2), \quad E_3 = \frac{1}{s\sqrt{2}}(e_3, -re_3),$$
$$E_4 = \frac{1}{r\sqrt{s^2+t^2}}(e_1, -r^2e_1), \quad E_5 = \frac{1}{s\sqrt{2}}(e_2, re_2), \quad E_6 = \frac{1}{s\sqrt{2}}(e_3, re_3)$$

We note that $\mathcal{F}_e$ or $\mathcal{G}_e$ is a subalgebra only when $r = 1$, in which case the relations (3.3) hold, and $P_e$ is a Lie algebra automorphism. [The symmetric Lie algebra $(\mathfrak{so}(3) \oplus \mathfrak{so}(3), \mathcal{F}_e, P_e)$ is isomorphic to the more usual $(\mathfrak{so}(3) \oplus \mathfrak{so}(3), \Delta\mathfrak{so}(3), \sigma)$ with $\sigma(X, Y) = (Y, X)$.] The corresponding invariant AP-structure $P$ is therefore harmonic (example 3.9). In general, the partial Ricci curvatures are given by (3.4), whose computation (a routine expansion of matrix brackets) yields

$$4\,\mathrm{Ric}_{\mathcal{F}}(V, B) = (1 - r^2)(ua + \tfrac{1}{2}vb + \tfrac{1}{2}wc),$$
$$4\,\mathrm{Ric}_{\mathcal{G}}(V, B) = (1 - r^2)(ua + \tfrac{3}{2}vb + \tfrac{3}{2}wc).$$

Therefore by theorem 3.1 $P$ is harmonic precisely when $r = 1$.

**Example 3.11.** Let $(M, g)$ be as in example 3.10, and let $(E_1, \ldots, E_6)$ be the following orthonormal basis of the Lie algebra:

$$E_1 = \frac{1}{s}(e_1, 0), \quad E_2 = \frac{1}{s}(e_3, 0), \quad E_3 = \frac{1}{t}(0, e_2),$$
$$E_4 = \frac{1}{s}(e_2, 0), \quad E_5 = \frac{1}{t}(0, e_1), \quad E_6 = \frac{1}{t}(0, e_3)$$

with $(e_1, e_2, e_3)$ given by (3.5). We define

$$\mathcal{F}_e = \mathrm{span}\{E_1, E_2, E_3\}, \quad \mathcal{G}_e = \mathrm{span}\{E_4, E_5, E_6\},$$

which satisfy the relations

$$0 \neq [\mathcal{F}, \mathcal{F}] \subset \mathcal{G}, \qquad 0 \neq [\mathcal{G}, \mathcal{G}] \subset \mathcal{F}.$$

The corresponding invariant AP-structure is therefore non-integrable. A simple computation of matrix brackets in (3.4) yields

$$\mathrm{Ric}_{\mathcal{F}}(V, B) = 0 = \mathrm{Ric}_{\mathcal{G}}(V, B)$$

and hence by theorem 3.1 this AP-structure is harmonic.

**Example 3.12.** Let $(M, g)$ be the tangent bundle of a Riemannian manifold $(M', g')$, equipped with the Sasaki metric. The foliation of $M$ by tangent spaces is a t.g. Riemannian foliation [16]. Let $\mathcal{F}$ be the vertical distribution; then $\mathcal{G}$ is the Levi-Civita horizontal distribution. Using [9, Thm. 1] it is easy to compute the relevant piece of the Ricci curvature: if $x \in M$ and $y, z \in M_x$ (the tangent space containing $x$) then

$$\mathrm{Ric}\big(y^{\mathcal{F}}(x), z^{\mathcal{G}}(x)\big) = \tfrac{1}{2}\,\delta' R'(z)(x,y)$$

where $y^{\mathcal{F}}(x) \in \mathcal{F}_x$ (resp. $z^{\mathcal{G}}(x) \in \mathcal{G}_x$) is the vertical lift of $y$ (resp. horizontal lift of $z$). By 3.2 therefore, this AP-structure is harmonic if and only if $(M', g')$ has harmonic curvature: $\delta' R' = 0$ (e.g. if $(M', g')$ is an Einstein manifold of dimension 3 or more; see also [4, Ch. 16]). From [9] it also follows that

$$\mathrm{Ric}\big(y^{\mathcal{F}}(x), z^{\mathcal{F}}(x)\big) = \tfrac{1}{4}\, g'\big(R'(x,y), R'(x,z)\big)$$

and so $(M, g)$ is Einstein if and only if $(M', g')$ is flat. The integrability tensor of $\mathcal{G}$ is

$$N_{\mathcal{G}}\big(y^{\mathcal{G}}(x), x^{\mathcal{G}}(x)\big) = -\tfrac{1}{2}\,\big(R(y,z)x\big)^{\mathcal{F}}$$

Therefore if $\dim M' = 4$ then $\mathcal{G}$ is $\pm$self-dual if and only if $R$ is $\pm$self-dual, if and only if $(M', g')$ is Ricci-flat and conformally half-flat [3].

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